Bayesian Estimators Assignment

Naveen Narayanan Meyyappan

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## Bayesian Estimators

In estimation theory and decision theory, a Bayes estimator or a Bayes action is an estimator or decision rule that minimizes the posterior expected value of a loss function (i.e., the posterior expected loss). Equivalently, it maximizes the posterior expectation of a utility function. An alternative way of formulating an estimator within Bayesian statistics is maximum a posteriori estimation.

## Question

Construct Bayesian estimates using the conjugate priors for each of the method of moments estimates Add it to the same program 10 points extra credit if you create a Bayesian calculation for the uniform parameters using MCMC and the bivariate normal distribution as a prior.

## Answer

In Bayesian probability theory, if the posterior distributions p(θ | x) are in the same probability distribution family as the prior probability distribution p(θ), the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function.

Conjugate families for samples from various standard distribution

Suppose that X1,…,Xn is a random sample from a Poisson distribution with an unknown value of the mean W. Suppose also that the prior distribution of W is a Gamma distribution with parameters alpha and beta such alpha>0 and beta>0. Then the posterior distribution of W when Xi=xi(i=1,…,n) is a Gamma distribution with parameters alpha + sum(xi) and beta + n

posterior\_poisson<-function(n,alpha,beta)  
{  
 priordist=rgamma(n,alpha,beta)# Generating the prior distribution  
 mu=mean(priordist)  
 variance=var(priordist)  
 beta\_hat=mu/variance  
 alpha\_hat=mu\*beta\_hat# Calculating alpha\_hat and Beta \_hat using the method of moments estimaators  
 lambda=alpha\_hat\*beta\_hat# calculating the lambda parameter for Poisson distribution  
 data=rpois(n,lambda)  
 alpha\_new=alpha+sum(data)  
 beta\_new=beta+n # Generating the parameters alpha and beta for the posterior Gamma distribution  
 return(list(prior\_distribution="Gamma", posterior\_distribution="Gamma",alpha\_new=alpha\_new,beta\_new=beta\_new))  
}  
posterior\_poisson(100,1,1)

## $prior\_distribution  
## [1] "Gamma"  
##   
## $posterior\_distribution  
## [1] "Gamma"  
##   
## $alpha\_new  
## [1] 86  
##   
## $beta\_new  
## [1] 101

Suppose that X1,…,Xn is a random sample from a Negative Binomial distribution with parameters r and W, where r has a specific value (r>0) and the value of W is unknown. Suppose also that the prior distribution of W is a Beta distribution with parameters alpha and beta such that alpha>0 and beta>0. Then the posterior distribution of W when Xi=xi(i=1,…,n) is a Beta distribution with parameters alpha + rn and beta + sum(xi)

posterior\_negativebinomial<-function(n,r,alpha,beta)  
{  
 priordist=rbeta(n,alpha,beta) # generating the prior beta distribution with the given alpha and beta values  
 mu=mean(priordist)  
 variance=var(priordist)  
 alpha\_hat = (((mu^2)-(mu^3)-(variance\*mu))/variance)  
 beta\_hat = (alpha\_hat\*(1-mu))/mu # Identifying alpha\_hat and beta\_hat using Method of moments estimators  
 w=(alpha\_hat/(alpha\_hat+beta\_hat))  
 data=rnbinom(n,r,w) # Calculating the parameter W for the distribution  
 alpha\_new=alpha\_hat+r\*n  
 beta\_new=beta\_hat+sum(data) # Generating the posterior distribution  
 return(list(prior\_distribution="Beta",posterior\_distribution="Beta",alpha\_new=alpha\_new,beta\_new=beta\_new))  
}  
posterior\_negativebinomial(100,0.5,1,2)

## $prior\_distribution  
## [1] "Beta"  
##   
## $posterior\_distribution  
## [1] "Beta"  
##   
## $alpha\_new  
## [1] 50.95996  
##   
## $beta\_new  
## [1] 83.95004

Suppose that X1,…,Xn is a random sample from an Exponential distribution with an unknown value of the parameter W. Suppose also that the prior distribution of W is a Gamma distribution with parameters alpha and beta such alpha>0 and beta>0. Then the posterior distribution of W when Xi=xi(i=1,…,n) is a Gamma distribution with parameters alpha + rn and beta + sum(xi)

posterior\_exponential<-function(n,alpha,beta)  
{  
 priordist=rgamma(n,alpha,beta)  
 mu=mean(priordist)  
 variance=var(priordist)  
 beta\_hat=mu/variance  
 alpha\_hat=mu\*beta\_hat  
 lambda=alpha\_hat\*beta\_hat  
 data=rexp(n,lambda)  
 alpha\_new=alpha+n  
 beta\_new=beta+sum(data)  
 return(list(prior\_distribution="Gamma", posterior\_distribution="Gamma", alpha\_new=alpha\_new, beta\_new=beta\_new))  
}  
posterior\_exponential(100,0.1,0.1)

## $prior\_distribution  
## [1] "Gamma"  
##   
## $posterior\_distribution  
## [1] "Gamma"  
##   
## $alpha\_new  
## [1] 100.1  
##   
## $beta\_new  
## [1] 58561

The Beta distribution is the conjugate prior for binomial observations. So if ones beliefs about a probability for a binomial experiment can be represented by a Beta distribution, then the conditional probability distribution for P after the experiment is also a Beta distribution

posterior\_binomial<-function(n,alpha,beta,size)  
{  
 priordist=rbeta(n,alpha,beta) # generating the prior distribution  
 mu=mean(priordist)  
 variance=var(priordist)  
 alpha\_hat = (((mu^2)-(mu^3)-(variance\*mu))/variance)  
 beta\_hat = (alpha\_hat\*(1-mu))/mu # Calculating alpha and beta using method of moments  
 w=(alpha\_hat/(alpha\_hat+beta\_hat))  
 data=rbinom(n,size,w)  
 alpha\_new=alpha\_hat+n # Computing the paramters for posterior distribution  
 beta\_new=beta\_hat+sum(data)  
 return(list(prior\_distribution="Beta",posterior\_distribution="Beta",alpha\_new=alpha\_new,beta\_new=beta\_new))  
}  
posterior\_binomial(100,2,3,10)

## $prior\_distribution  
## [1] "Beta"  
##   
## $posterior\_distribution  
## [1] "Beta"  
##   
## $alpha\_new  
## [1] 102.5045  
##   
## $beta\_new  
## [1] 406.9228

Suppose that X1,…,Xn is a random sample from an Normal distribution with an unknown value of the mean M and an unknown value of precision R. Suppose also that the prior joint distribution of M and R is as follows: The conditional distribution of M when R=r (r>0) is a normal distribution with mean mu and precision tau such that -infinity<mu<+infinity and r>0,and the marginal distribution of R is a Gamma distribution with parameters alpha and beta such alpha>0 and beta>0. Then the posterior joint distribution of M and R when Xi=xi(i=1,…,n) is as follows with mean mu’ and precision (r+n)r, and the marginal distribution of R is a Gamma distribution with parameters alpha + n/2 and beta’

posterior\_normal\_mu\_unknown\_precision\_unknown<-function(n,mu,tau,alpha,beta)  
{  
 priordist=rnorm(n,mu,1/tau)#prior dist 1 is for mean M  
 mu\_hat=mean(priordist)  
 sigma\_hat=sd(priordist)  
 tau\_hat=1/sigma\_hat # Calculating the parameters using method of moments  
 priordist2=rgamma(n,alpha,beta) #prior dist 2 for precision r  
 mu\_g=mean(priordist2)  
 variance\_g=var(priordist2)  
 beta\_hat=mu\_g/variance\_g  
 alpha\_hat=mu\_g\*beta\_hat # Calculating the parameters using method of moments  
 r=alpha\_hat/(alpha\_hat+beta\_hat)  
 data=rnorm(n,mu\_hat,1/r) # Generating the distribution  
 mu\_new=((tau\_hat\*mu\_hat) + (sum(data)))/(tau\_hat + n)  
 precision\_new=(tau\_hat+n)\*r  
 alpha\_new=alpha\_hat+(n/2) #Identifying the parameters for the posterior distributions  
 beta\_new=beta\_hat+((0.5\*variance\_g+tau\_hat\*(mu\_g-mu\_hat))/(2\*tau\_hat\*n))  
 return(list(prior\_distribution\_of\_M="Normal",posterior\_distribution\_of\_M="Normal",mu=mu\_new, precision=precision\_new, prior\_distibution\_R="Gamma",posterior\_distribution\_R="Gamma",alpha\_new=alpha\_new,beta\_new=beta\_new))  
}  
posterior\_normal\_mu\_unknown\_precision\_unknown(100,1,1,1,1)

## $prior\_distribution\_of\_M  
## [1] "Normal"  
##   
## $posterior\_distribution\_of\_M  
## [1] "Normal"  
##   
## $mu  
## [1] 0.9325971  
##   
## $precision  
## [1] 49.33475  
##   
## $prior\_distibution\_R  
## [1] "Gamma"  
##   
## $posterior\_distribution\_R  
## [1] "Gamma"  
##   
## $alpha\_new  
## [1] 51.0404  
##   
## $beta\_new  
## [1] 1.091755

## References

1. <https://en.wikipedia.org/wiki/Bayes_estimator>
2. <https://en.wikipedia.org/wiki/Conjugate_prior>
3. <https://www.johndcook.com/blog/conjugate_prior_diagram/>